

ON THE JUMPING PHENOMENON OF $\dim_{\mathbb{C}} H^q(X_t, E_t)$

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ABSTRACT. Let X be a compact complex manifold and E be a holomorphic vector bundle on X . Given a deformation $(\mathcal{X}, \mathcal{E})$ of the pair (X, E) over a small polydisk B centered at the origin, we study the jumping phenomenon of the cohomology groups $\dim_{\mathbb{C}} H^q(X_t, E_t)$ near $t = 0$. We show that there are precisely two cohomological obstructions to the stability of $\dim_{\mathbb{C}} H^q(X_t, E_t)$, which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye [7, 8].

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1. INTRODUCTION

Let X be a compact complex manifold and $\pi : \mathcal{X} \rightarrow B$ be a small deformation of $X = \pi^{-1}(0)$ over a small polydisk B centered at the origin in some complex vector space. Suppose that \mathcal{F} is a coherent sheaf on \mathcal{X} which is flat over B . Then the sheaf \mathcal{F} can be viewed as a deformation of the sheaf $\mathcal{F}|_X$ on X . It is known by Grauert's direct image theorem that the dimension $\dim_{\mathbb{C}} H^q(\mathcal{X}_t, \mathcal{F}_t)$ is an upper semi-continuous function in t . Moreover, we have the following characterization for when the dimension $\dim_{\mathbb{C}} H^q(\mathcal{X}_t, \mathcal{E}_t)$ is locally constant, also due to Grauert.

Theorem 1.1 (Grauert [2]). *Let $\pi : \mathcal{X} \rightarrow B$ be a flat proper holomorphic map between complex analytic space \mathcal{X}, B with B being reduced and connected. Suppose that \mathcal{F} is a coherent sheaf on \mathcal{X} that is flat over B . Let $k(t) := \mathcal{O}_{B,t}/\mathfrak{m}_t$ be the residue field at $t \in B$ and \mathcal{F}_t be the pullback of \mathcal{F} to \mathcal{X}_t . Then the following are equivalent:*

(a) *The function*

$$t \mapsto \dim_{\mathbb{C}} H^q(\mathcal{X}_t, \mathcal{F}_t)$$

is locally constant in $t \in B$.

(b) *The sheaf $R^q\pi_*\mathcal{F}$ is locally free and the natural map*

$$R^q\pi_*\mathcal{F} \otimes k(t) \rightarrow H^q(\mathcal{X}_t, \mathcal{F}_t)$$

is an isomorphism.

Nevertheless, condition (b) in the above theorem is not easy to check in general even when \mathcal{F} is locally free. In [7, 8], X. Ye studied the jumping phenomenon of the dimensions $\dim_{\mathbb{C}} H^q(X, \bullet)$ under small deformations of X , where $\bullet = \Omega_X^p, T_X$. He found two obstructions $o_{n,n-1}^q, o_{n,n-1}^{q-1}$ and proved that the dimension of $H^q(X, \bullet)$ does not jump if and only if $o_{n,n-1}^q \equiv 0, o_{m,m-1}^{q-1} \equiv 0$ for all $n, m \geq 1$.

In this note, we generalize Ye's results to a much more general setting, namely, when X is a compact complex manifold and E is an arbitrary holomorphic vector bundle on X . Let $(\mathcal{X}, \mathcal{E})$ be a small deformation of (X, E) over a polydisk B centered at the origin. We assume that \mathcal{E} is flat over B via the proper holomorphic submersion $\pi : \mathcal{X} \rightarrow B$. Let $\mathcal{E}_t := \mathcal{E}|_{\mathcal{X}_t}$. We are interested in characterizing when the dimension $\dim_{\mathbb{C}} H^q(\mathcal{X}_t, \mathcal{E}_t)$ stays constant near $t = 0$.

Following [7, 8], we formulate the jumping phenomenon of $\dim_{\mathbb{C}} H^q(\mathcal{X}_t, \mathcal{E}_t)$ as an extension problem, namely, whether we can extend a nonzero element in $H^q(X, E)$ to one in a nearby fiber $H^q(\mathcal{X}_t, \mathcal{E}_t)$. In general, such extensions may not exist, and there are obstructions to this extension problem. In [7, 8], Ye gave explicit formulae for these obstructions in the cases when $\mathcal{E} = \Omega_{\mathcal{X}/B}^p$ and $\mathcal{E} = T_{\mathcal{X}/B}$. We will see that his formulae can be generalized to the above general setting.

While Ye [7, 8] employed an algebraic approach to study the extension problem by applying a version of Grauert's direct image theorem which states that $R^q\pi_*\mathcal{E}$ is a quotient of two locally free sheaves of finite ranks over B , here we adapt a differential-geometric approach, following [4, 1].

We will formulate the problem directly as extending E -valued differential forms over B , which means that, in contrary to [7, 8], we are going to work with sheaves of infinite rank. A key step is to get an explicit description of $R^q\pi_*\mathcal{E}$ by using an acyclic resolution $(\mathcal{D}^\bullet, \bar{D}^\bullet)$ of the sheaf \mathcal{E} constructed from the differential operators \bar{D}^\bullet studied in [4, 1] (see Section 3). The operators \bar{D}^\bullet capture the holomorphic structures of the deformed pairs $\{(\mathcal{X}_t, \mathcal{E}_t)\}_{t \in B}$ (see [1] and also Section 2). Then more or less the same strategy as in Ye's proofs will work. An advantage of this geometric approach is that the computation of the obstructions becomes much simpler and more transparent, as compared to the Čech calculations in [7, 8]. Our main result is as follows (see Section 4, in particular, Theorem 4.11 and Equations (1) & (2) for the details):

Theorem 1.2. *Let $\{(A(t), \varphi(t))\}_{t \in B}$ be the family of Maurer-Cartan elements associated to the small deformation $(\mathcal{X}, \mathcal{E})$ of (X, E) . We define the n -th order obstruction maps $O_{n,n-1}^i : H^i((\pi_*\mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n) \rightarrow H^{i+1}((\pi_*\mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$, where $i = q, q-1$, by*

$$O_{n,n-1}^i([\alpha_{n-1}]) = \left[t^{n-1} \sum_{j=0}^{n-1} (\varphi^{n-j} \lrcorner \nabla + A^{n-j}) \alpha_{n-1}^j \right].$$

Then the function $t \mapsto \dim_{\mathbb{C}} H^q(\mathcal{X}_t, \mathcal{E}_t)$ is locally constant if and only if $O_{m,m-1}^q \equiv 0$ and $O_{n,n-1}^{q-1} \equiv 0$ for all $m, n \geq 1$.

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2. DEFORMATIONS OF PAIRS

In this section, we briefly review the deformation theory of a pair (X, E) , where X is a compact complex manifold and E is a holomorphic vector bundle on X , following the exposition in [1] (cf. [4]), and recall several useful facts.

Definition 2.1. *Let B be a small polydisk in some complex vector space containing the origin. A deformation of (X, E) consists of a surjective proper submersion $\pi : \mathcal{X} \rightarrow B$ between complex manifolds \mathcal{X} and B , together with a holomorphic vector bundle \mathcal{E} on \mathcal{X} , such that $\pi^{-1}(0) = X$ and $\mathcal{E}|_{\pi^{-1}(0)} = E$.*

Given such a deformation of (X, E) , we put $\mathcal{X}_t := \pi^{-1}(t)$ and $\mathcal{E}_t := \mathcal{E}|_{\mathcal{X}_t}$. Since we have assumed that B is a polydisk, it is contractible and thus we have $\mathcal{X} \cong X \times B$ as smooth manifolds and $\mathcal{E} \cong E \times B$ as smooth vector bundles. Let $\varphi(t) \in \Omega^{0,1}(T_X)$ be the family of Maurer-Cartan elements which corresponds to the family $\mathcal{X} \rightarrow B$. In [1], we considered a holomorphic family of differential operators $\bar{D}_t^q : \Omega^{0,q}(E) \rightarrow \Omega^{0,q+1}(E)$ defined locally by

$$\bar{D}_t \left(\sum_j \alpha_j \otimes e_j(t) \right) := \sum_j (\bar{\partial} + \varphi(t) \lrcorner \partial) \alpha_j \otimes e_j(t),$$

where $\{e_j(t)\}$ are local holomorphic frames of \mathcal{E}_t .

By choosing a Hermitian metric on E and the Chern connection ∇ , one can express \bar{D}_t as

$$\bar{D}_t^q = \bar{\partial}_E + \varphi(t) \lrcorner \nabla + A(t),$$

for some $A(t) \in \Omega^{0,1}(\text{End}(E))$. Then $(A(t), \varphi(t)) \in \Omega^{0,1}(A(E))$ is the family of Maurer-Cartan elements which corresponds to the deformation $(\mathcal{X}, \mathcal{E})$, namely, we have

$$\bar{\partial}_{A(E)}(A(t), \varphi(t)) + \frac{1}{2}[(A(t), \varphi(t)), (A(t), \varphi(t))] = 0$$

for $t \in B$; here $A(E)$ is the Atiyah extension of E . This family of operators satisfies the integrability condition $\bar{D}_t^q \bar{D}_t^{q-1} = 0$ (which is equivalent to the Maurer-Cartan equation).

Another important feature of the operator \bar{D}_t , which will be useful later, is that its cohomology computes the Dolbeault cohomology of $(\mathcal{X}_t, \mathcal{E}_t)$.

Proposition 2.2 ([1], Proposition 3.13). *For each fixed $t \in B$, we have*

$$H^q(\mathcal{X}_t, \mathcal{E}_t) \cong H^q((\pi_* \mathcal{D}^\bullet)_t \otimes k(t)) \cong H^q(\Omega^{0,\bullet}(E), \bar{D}_t),$$

for any $q \geq 0$.

3. AN ACYCLIC RESOLUTION FOR \mathcal{E}

From now on, for the purpose of simplifying computations and formulae, we will assume that the base B of the deformation is complex 1-dimensional. In this section, we will construct an acyclic resolution of the sheaf \mathcal{E} in order to get an explicit description of the direct image sheaf $R^q\pi_*\mathcal{E}$.

Let $\pi_X : \mathcal{X} \rightarrow X$ be the projection (not necessary holomorphic) of \mathcal{X} onto X . Consider the sheaf of \mathcal{O}_X -modules \mathcal{D}^q over \mathcal{X} defined by

$$\mathcal{D}^q : \mathcal{U} \mapsto \{\alpha \in \Gamma_{\text{smooth}}(\mathcal{U}, \pi_X^* \Omega_X^{0,p} \otimes \mathcal{E}) : \bar{\partial}_B \alpha = 0\},$$

whose pushforward by π carries a natural \mathcal{O}_B -module structure. Since \bar{D}_t varies holomorphically in the variable t , it induces a sheaf map $\bar{D}^q : \mathcal{D}^q \rightarrow \mathcal{D}^{q+1}$ for each $q \geq 0$. We further define $\tilde{\mathcal{D}}^{q,\bullet}$ to be the sheaf of smooth sections of $\pi_B^* \Omega_B^{0,\bullet} \otimes \pi_X^* \Omega_X^{0,p} \otimes \mathcal{E}$. Clearly, $\mathcal{D}^\bullet \subset \tilde{\mathcal{D}}^{\bullet,0}$ as \mathcal{O}_X -submodules. Let $\bar{\partial}_B$ be the Dolbeault operator on the base B . Then $\bar{\partial}_B$ gives a complex $(\tilde{\mathcal{D}}^{q,\bullet}, \bar{\partial}_B^\bullet)$ for each q .

Lemma 3.1. *For each $p, q \geq 0$ the sheaf $\tilde{\mathcal{D}}^{q,p}$ is fine and the complex $(\pi_* \tilde{\mathcal{D}}^{q,\bullet}, \pi_* \bar{\partial}_B^\bullet)$ has no higher cohomology sheaves, that is,*

$$\mathcal{H}^p(\pi_* \tilde{\mathcal{D}}^{q,\bullet}) = 0$$

for all $p \geq 1$.

Proof. Fineness is clear, for we can apply a partition of unity to conclude that $\tilde{\mathcal{D}}^{q,p}$ has no higher direct images.

To prove that $(\pi_* \tilde{\mathcal{D}}^{q,\bullet}, \pi_* \bar{\partial}_B^\bullet)$ has no higher cohomology, we recall that $\mathcal{H}^p(\pi_* \tilde{\mathcal{D}}^{q,\bullet})$ is the sheafification of

$$W \mapsto H^p(\Gamma(\pi^{-1}(W), \tilde{\mathcal{D}}^{q,\bullet})).$$

It suffices to prove that $H^p(\Gamma(\pi^{-1}(W), \tilde{\mathcal{D}}^{q,\bullet})) = 0$ for any polydisk $W \subset B$ and all $p \geq 1$. Let $\alpha \in \Gamma(\pi^{-1}(W), \tilde{\mathcal{D}}^{q,p})$ and $\{U_i\}$ be a locally finite open covering of $X \subset \pi^{-1}(W)$ by coordinates charts. Let α_i be the restriction of α on $U_i \times W$. Write

$$\alpha_i = \sum_{I,J} \alpha_{IJ,i}(z, \bar{z}, t, \bar{t}) dt^J \otimes d\bar{z}^I =: \sum_I \alpha_{I,i} \otimes d\bar{z}^I.$$

Then $\bar{\partial}_B \alpha = 0$ simply means for each I ,

$$0 = \bar{\partial}_B \left(\sum_J \alpha_{IJ,i} dt^J \right) = \bar{\partial}_B \alpha_{I,i}.$$

Hence, for fixed z , we can apply the Dolbeault lemma on W to conclude that

$$\alpha_{I,i} = \bar{\partial}_B \beta_{I,i},$$

for some $\beta_{I,i} \in \Omega_B^{0,p-1}(W)$. Since $\alpha_{I,i}$ varies smoothly in z and \bar{z} , we see from the proof of the Dolbeault-Grothendieck lemma that $\beta_{I,i}$ can be chosen to be smooth in z as well. Let $\{\psi_i\}$ be a partition of unity on X subordinate to the covering $\{U_i\}$. Define

$$\beta := \sum_{I,i} \psi_i \beta_{I,i} \otimes d\bar{z}^I.$$

Then $\beta \in \Gamma(\pi^{-1}(W), \tilde{\mathcal{D}}^{q,p-1})$ and

$$\bar{\partial}_B \beta = \sum_{I,i} \psi_i (\bar{\partial}_B \beta_{I,i}) \otimes d\bar{z}^I = \sum_{I,i} \psi_i \alpha_{I,i} \otimes d\bar{z}^I = \left(\sum_i \psi_i \right) \alpha = \alpha.$$

We have the first equality simply because $\{\psi_i\}$ are all independent of t and \bar{t} , and the second last equality follows from the fact that α is a global section on $\pi^{-1}(W)$. \square

Lemma 3.2. *For each $q \geq 0$, the sheaf \mathcal{D}^q is acyclic with respect to the left-exact functor π_* .*

Proof. Lemma 3.1 shows that $(\tilde{\mathcal{D}}^{q,\bullet}, \bar{\partial}_B^\bullet)$ is a fine resolution of \mathcal{D}^q and so $R^q \pi_* \mathcal{D}^q \cong \mathcal{H}^p(\pi_* \tilde{\mathcal{D}}^{q,\bullet}) = 0$ for all $q \geq 1$. \square

Proposition 3.3. *The complex of sheaves $(\mathcal{D}^\bullet, \bar{D}^\bullet)$ is an acyclic resolution of \mathcal{E} with respect to the left-exact functor π_* . In particular, we have*

$$R^q \pi_* \mathcal{E} \cong \mathcal{H}^q(\pi_* \mathcal{D}^\bullet)$$

as \mathcal{O}_B -modules.

Proof. By Lemma 3.2, $R^p \pi_* \mathcal{D}^q = 0$ for all $p \geq 1$. It remains to prove that it defines a resolution of \mathcal{E} . We need to show that for any point $(p, t) \in \mathcal{X} \cong X \times B$, the sequence of stalks

$$0 \rightarrow \mathcal{E}_{(p,t)} \rightarrow \mathcal{D}_{(p,t)}^0 \rightarrow \mathcal{D}_{(p,t)}^1 \rightarrow \cdots$$

is exact. The exactness of

$$0 \rightarrow \mathcal{E}_{(p,t)} \rightarrow \mathcal{D}_{(p,t)}^0 \rightarrow \mathcal{D}_{(p,t)}^1$$

follows from the fact that \bar{D}^0 and $\bar{\partial}_{\mathcal{E}}$ share the same kernel. For the remaining exactness, we will focus on the case $t = 0$; same argument works for general $t \in B$.

Recall that \bar{D}_t is locally defined by

$$\bar{D}_t \left(\sum_j \alpha_j \otimes e_j(t) \right) := \sum_j (\bar{\partial} + \varphi(t) \lrcorner \partial) \alpha_j \otimes e_j(t),$$

so it suffices to prove the exactness for the case $\mathcal{E} = \mathcal{O}_{\mathcal{X}}$.

We would like to first work over $\mathbb{C}[[t]]$ instead of $\mathbb{C}\{t\}$. Let $U \subset X$ be a polydisk and denote $\Omega^{0,\bullet}(U)\{t\} := \Omega^{0,\bullet}(U) \otimes_{\mathbb{C}} \mathbb{C}\{t\}$ and $\Omega^{0,\bullet}(U)[[t]] := \Omega^{0,\bullet}(U) \otimes_{\mathbb{C}} \mathbb{C}[[t]] = \Omega^{0,\bullet}(U)\{t\} \otimes_{\mathbb{C}\{t\}} \mathbb{C}[[t]]$. The Maurer-Cartan element $\varphi(t)$ is gauge equivalent to 0 on U . Hence

$$\bar{\partial} + \varphi(t) \lrcorner \partial = e^{v(t)} \bar{\partial} e^{-v(t)},$$

for some $v(t) \in \Omega^0(T_U)[[t]]$ and $e^{v(t)}$ acts on $\Omega^{0,q}(U)[[t]]$ by

$$e^{v(t)} \alpha(t) = \sum_{n=0}^{\infty} \frac{(v(t) \lrcorner \partial)^n}{n!} \alpha(t).$$

Hence we can apply the Dolbeault-Grothendieck lemma with analytic parameter (the t -variable) to conclude that $(\Omega^{0,\bullet}(U)[[t]], \bar{D}_t^\bullet)$ is an exact complex.

Now, as $\mathbb{C}[[t]]$ is a flat- $\mathbb{C}\{t\}$ module (because $\mathbb{C}[[t]]$ is torsion free and $\mathbb{C}\{t\}$ is a PID), we have

$$H^q(\Omega^{0,\bullet}(U)[[t]]) = H^q(\Omega^{0,\bullet}(U)\{t\} \otimes \mathbb{C}[[t]]) \cong H^q(\Omega^{0,\bullet}(U)\{t\}) \otimes \mathbb{C}[[t]].$$

But we have shown that $H^q(\Omega^{0,\bullet}(U)[[t]]) = 0$. Therefore, $H^q(\Omega^{0,\bullet}(U)\{t\}) \otimes \mathbb{C}[[t]] = 0$. If we can show that $H^q(\Omega^{0,\bullet}(U)\{t\})$ is torsion free, we see that $H^q(\Omega^{0,\bullet}(U)\{t\})$ vanishes. Assuming this, we conclude that every \bar{D}_t -closed $(0, q)$ -form valued power series on U is locally exact.

Now, for any $\bar{D}_{(p,0)}$ -closed element $\alpha \in \mathcal{D}_{(p,0)}^q$, we can represent it by a \bar{D}_t -closed element $\alpha(t) \in \Omega^{0,q}(U)\{t\}$, for some polydisk $U \subset X$. The vanishing of $H^q(\Omega^{0,\bullet}(U)\{t\})$ shows that $\alpha(t) = \bar{D}_t \beta(t)$ for some $\beta(t) \in \Omega^{0,q-1}(U)\{t\}$. This $\beta(t)$ defines an element $\beta \in \mathcal{D}_{(p,0)}^{q-1}$ such that $\bar{D}^{q-1} \beta = \alpha$. This proves the exactness of the complex $(\mathcal{D}^\bullet, \bar{D}^\bullet)$.

To complete the proof of the proposition, we need to prove that $H^q(\Omega^{0,\bullet}(U)\{t\})$ is a torsion free $\mathbb{C}\{t\}$ -module for $q > 1$. That is, if $[\alpha(t)] \in H^q(\Omega^{0,\bullet}(U)\{t\})$ is a nonzero element, then $f(t) \cdot [\alpha(t)]$ is nonzero for all $f(t) \in \mathbb{C}\{t\} - \{0\}$. Since $f(t)$ is invertible if $f(0) \neq 0$, we may assume $f(t) \in (t^N)$ for some $N \geq 1$. We may assume N is chosen such that $f(t) = t^N g(t)$ with $g(0) \neq 0$. Again, we can invert $g(t)$, so we can further assume $f(t) = t^N$. Then the vanishing of $f(t) \cdot [\alpha(t)] = [f(t) \cdot \alpha(t)]$ means

$$t^N \alpha(t) = \bar{D}_t \beta(t) = (\bar{\partial} + \varphi(t) \lrcorner \partial) \beta(t),$$

for some $\beta(t) \in \Omega^{0,q-1}(U)\{t\}$. Since both $\alpha(t)$ and $\beta(t)$ is holomorphic in t , the equation shows that $\beta(t)$ is in fact \bar{D}_t -closed up to order $N - 1$.

We first prove the following

Lemma 3.4. *For any $\bar{\partial}$ -closed $\beta \in \Omega^{0,q-1}(U)$, $q > 1$, there exists $\beta(t) \in \Omega^{0,q-1}(U)\{t\}$ such that*

$$\beta(0) = \beta \text{ and } \bar{D}_t \beta(t) = 0.$$

Proof of Lemma 3.4. Since β is $\bar{\partial}$ -closed on the polydisk U , it must be $\bar{\partial}$ -exact. Write $\beta = \bar{\partial} \alpha$ for some $\alpha \in \Omega^{0,q-2}(U)$. Define

$$\beta(t) := \beta + \varphi(t) \lrcorner \partial \alpha \in \Omega^{0,q-1}(U)\{t\}.$$

Then $\beta(0) = \beta$. Since $\bar{D}_t^2 = 0$, we have $\bar{D}_t \beta(t) = 0$. \square

With this lemma in hand, we see that $\alpha(t)$ is \bar{D}_t -exact and this proves that $H^q(\Omega^{0,\bullet}(U)\{t\})$ is torsion free.

Since β_0 is $\bar{\partial}$ -closed, we can choose $\beta_1(t) \in \Omega^{0,q-1}(U)\{t\}$ such that

$$\beta_1(0) = \beta_0 \text{ and } \bar{D}_t \beta_1(t) = 0.$$

Then we have

$$t^{N-1} \alpha(t) = \bar{D}_t \left(\frac{\beta(t) - \beta_1(t)}{t} \right) = \bar{D}_t \gamma_1(t).$$

If $N = 1$, we are done. Otherwise, we see that $\gamma_1(0)$ is $\bar{\partial}$ -closed. Hence we can find $\beta_2(t)$ such that

$$\beta_2(0) = \gamma_1(0) \text{ and } \bar{D}_t \beta_2(t) = 0.$$

Hence

$$t^{N-2} \alpha(t) = \bar{D}_t \left(\frac{\gamma_1(t) - \beta_2(t)}{t} \right).$$

Repeating this process, we will arrive at the conclusion that

$$\alpha(t) = \bar{D}_t \gamma_N(t),$$

for some $\gamma_N(t) \in \Omega^{0,q-1}(U)\{t\}$. This completes the proof of the proposition. \square

4. OBSTRUCTIONS

In this section, we will find out explicitly the obstruction maps for extending a given element of $H^q(X, E)$. In [7, 8], the author used the Grauert direct image theorem to obtain a complex of locally free \mathcal{O}_B -modules of finite ranks to compute the obstruction maps, while we use the infinite-dimensional complex of \mathcal{O}_B -modules $(\pi_* \mathcal{D}^\bullet, \bar{D}^\bullet)$. We are going to see that more or less the same strategy of proofs in [7, 8] will work in our infinite-dimensional setting as well. We give most details of the proofs in order to be more self-contained.

Recall that Proposition 3.3 gives an isomorphism of \mathcal{O}_B -modules:

$$R^q \pi_* \mathcal{E} \cong \mathcal{H}^q(\pi_* \mathcal{D}^\bullet).$$

Together with Proposition 2.2, we see that it is equivalent to work with the sheaf $\mathcal{H}^q(\pi_* \mathcal{D}^\bullet)$ and the cohomology group $H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0))$. Tensoring the stalk $(\pi_* \mathcal{D}^\bullet)_0$ with $\mathcal{O}_{B,0}/m_0^{n+1}$ over $\mathcal{O}_{B,0}$, we obtain a complex

$$((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^{n+1}, \bar{D}_n^\bullet).$$

Given $\alpha \in \ker(\bar{\partial}_E^q)$, and suppose that we have a local extension $\alpha_{n-1} \in \Gamma(U, \pi_* \mathcal{D}^q)$ of α such that

$$j_0^{n-1}(\bar{D}^q \alpha_{n-1})(t) = 0,$$

we define the obstruction map $O_{n,n-1}^q : H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n) \rightarrow H^{q+1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$ by

$$(1) \quad O_{n,n-1}^q[j_0^{n-1}(\alpha_{n-1})(t)] := [t^{n-1} \cdot (j_0^n(\bar{D}^q \alpha_{n-1})(t)/t^n)]$$

Remark 4.1. *The $(n-1)$ -st jet can be viewed as an element in $(\pi_* \mathcal{D}^q)_0 \otimes \mathcal{O}_{B,0}/m_0^n$. The map $O_{n,n-1}^q$ factors through a map*

$$O_n^q : H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n) \rightarrow H^{q+1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n),$$

given by

$$O_n^q[j_0^{n-1}(\alpha_{n-1})(t)] := [j_0^q(\bar{D}^q \alpha_{n-1})(t)/t^n].$$

This is well-defined because the cohomology class of $j_0^q(d^q \alpha_{n-1})(t)/t^n$ only depends on the cohomology class of the $(n-1)$ -st jet $j_0^{n-1}(\alpha_{n-1})(t)$.

For later use, we define

$$O_{n,i}^q[j_0^{n-1}(\alpha_{n-1})(t)] := [t^i \cdot (j_0^q(\bar{D}^q \alpha_{n-1})(t)/t^n)],$$

for $i \geq 0$ and $n \geq 1$

The following proposition characterizes when an extension exists up to order $n \geq 1$.

Proposition 4.2. *For a fixed $n \geq 1$, the following are equivalent:*

- (1) *For any local section α_{n-1} around $t = 0$ such that $j_0^{n-1}(\bar{D}^q \alpha_{n-1})(t) = 0$, there exists a local section α_n around $t = 0$ such that $j_0^0(\alpha_n - \alpha_{n-1}) = 0$ and $j_0^n(\bar{D}^q \alpha_n)(t) = 0$.*
- (2) *For any $c_{n-1} \in H^q((\pi_* \mathcal{D}^q)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$, there exists $c_n \in H^q((\pi_* \mathcal{D}^q)_0 \otimes \mathcal{O}_{B,0}/m_0^{n+1})$ such that $c_n|_{t=0} = c_{n-1}|_{t=0} \in H^q((\pi_* \mathcal{D}^q)_0 \otimes k(0))$.*
- (3) *For any local section α_{n-1} around $t = 0$ such that $j_0^{n-1}(\bar{D}^q \alpha_{n-1})(t) = 0$, $O_{n,n-1}^q[j_0^{n-1}(\alpha_{n-1})(t)] = 0$.*

Proof. We shall prove that (1) \Leftrightarrow (2) and (1) \Leftrightarrow (3).

For (1) \Rightarrow (2) : Let $c_{n-1} \in H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$ and α_{n-1} be a local section around $t = 0$ such that $j_0^{n-1}(\alpha_{n-1})(t) \in \ker(\bar{D}_{n-1}^q)$ represents the class c_{n-1} . Then $j_0^{n-1}(\bar{D}^q \alpha_{n-1})(t) = 0$. By assumption, we can extend α_{n-1} to a local section α_n around $t = 0$ such that $j_0^0(\alpha_n - \alpha_{n-1})(t) = 0$ and $j_0^n(\bar{D}^q \alpha_n)(t) = 0$. Then $\bar{D}_n^q(j_0^n(\alpha_n)(t)) = 0 \in (\pi_* \mathcal{D}^{q+1})_0 \otimes \mathcal{O}_{B,0}/m_0^{n+1}$. Set $c_n := [j_0^n(\alpha_n)(t)] \in H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^{n+1})$. Since $j_0^0(\alpha_n - \alpha_{n-1}) = 0$, we have $c_n|_{t=0} = [j_0^0(\alpha_{n-1})(t)] = c_{n-1}|_{t=0} = 0$.

For (2) \Rightarrow (1) : Let α_{n-1} be such that $j_0^{n-1}(\bar{D}^q \alpha_{n-1})(t) = 0$. Extend $c_{n-1} := [j_0^{n-1}(\alpha_{n-1})(t)]$ to a class $c_n \in H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^{n+1})$. Let α_n be local section around $t = 0$ such that $j_0^n(\alpha_n)(t)$ represents the class c_n . Then $j_0^n(\bar{D}^q \alpha_n)(t) = 0$. Since $c_n|_{t=0} = c_{n-1}|_{t=0}$, we have

$$j_0^0(\alpha_n - \alpha_{n-1}) = \bar{D}_0^{q-1} \gamma,$$

for some $\gamma \in (\pi_* \mathcal{D}^{q-1})_0 \otimes k(0)$. Choose any representative γ' of γ and define

$$\alpha'_n := \alpha_n - \bar{D}^{q-1} \gamma'.$$

Then $j_0^n(\bar{D}^q \alpha'_n)(t) = j_0^n(\bar{D}^q \alpha_n)(t) = 0$ and $j_0^0(\alpha'_n - \alpha_{n-1}) = 0$.

For (1) \Rightarrow (3) : Let $\gamma := \alpha_{n-1} - \alpha_n$. Then

$$\bar{D}_n^q \gamma = j_0^n(\bar{D}^q(\alpha_{n-1} - \alpha_n))(t) = t^n \cdot (j_0^n(\bar{D}^q \alpha_{n-1})(t)/t^n),$$

since $j_0^{n-1}(\bar{D}^q \alpha_{n-1})(t) = j_0^n(d^q \alpha_n)(t) = 0$. By assumption, $j_0^0(\gamma)(t) = 0$, so $\gamma = t\beta$ for some local section β around $t = 0$. Hence

$$\bar{D}_{n-1}^q j_0^{n-1}(\beta)(t) = t^{n-1} \cdot (j_0^n(\bar{D}^q \alpha_{n-1})(t)/t^n),$$

which means that $O_{n,n-1}^q[j_0^{n-1}(\alpha_{n-1})(t)] = 0$.

For (3) \Rightarrow (1) : The vanishing of $O_{n,n-1}^q[j_0^{n-1}(\alpha_{n-1})(t)]$ gives an element $\beta \in (\pi_* \mathcal{D}^q)_0 \otimes \mathcal{O}_{B,0}/m_0^n$ such that

$$t^{n-1} \cdot (j_0^n(\bar{D}^q \alpha_{n-1})(t)/t^n) = \bar{D}_n^q \beta.$$

Let β' be a local section around $t = 0$ representing the germ β and set $\alpha_n := \alpha_{n-1} - t\beta'$. Then

$$j_0^n(\bar{D}^q \alpha_n)(t) = j_0^n(\bar{D}^q \alpha_{n-1})(t) - t \cdot j_0^{n-1}(\bar{D}^q \beta')(t) = t^n \cdot (j_0^n(\bar{D}^q \alpha_{n-1})(t)/t^n) - t \cdot \bar{D}_n^q \beta = 0.$$

Hence α_n defines an n -th order extension of α . \square

Therefore, if $O_{n,n-1}^q \equiv 0$ for all $n \geq 1$, then by (1) above we obtain a formal element $\alpha(t)$ such that $\bar{D}_t \alpha(t) = 0$. In Appendix A, we will show that after a gauge fixing, $\alpha(t)$ is analytic in a neighborhood around $0 \in B$.

Remark 4.3. *The radius of convergence of each extension $\alpha(t)$ may be different as $\alpha = \alpha(0)$ varies. However, since $H^q(X, E)$ is finite dimensional, we can simply choose a basis, for instance, consisting of harmonic forms with respect to a certain hermitian metric. Then we obtain a minimum radius of convergence, uniform in all $[\alpha] \in H^q(X, E)$.*

Next we shall demonstrate that there is another obstruction for an extension to be nonzero.

Proposition 4.4. *A non-exact element $\beta \in \ker(\bar{\partial}_E^q)$ admits a local extension $\beta(t) \in \Gamma(U, \pi_* \mathcal{D}^q)$ such that $\beta(t)$ is exact for $t \neq 0$ if and only if there exist $n \geq 1$ and $[j_0^{n-1}(\alpha_{n-1})(t)] \in H^{q-1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$ such that*

$$O_n^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] = [\beta].$$

Proof. Suppose that $O_n^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] = [\beta]$. Then

$$\beta = j_0^n(\bar{D}^{q-1}\alpha_{n-1})(t)/t^n + \bar{\partial}_E^{q-1}\gamma$$

for some $\gamma \in \Omega^{0,q-1}(E)$. Define $\beta(t)$ by

$$\beta(t) := \bar{D}^{q-1}(\alpha_{n-1}(t)/t^n) + \bar{D}^{q-1}\gamma(t), \quad t \neq 0,$$

where $\gamma(t)$ is any extension of γ . Clearly $\beta(t)$ can be extended through the origin by setting $\beta(0) = \beta$. Then $\beta(t)$ is a \bar{D}^{q-1} -exact class and equals β at $t = 0$. Hence $\beta(t)$ serves as an extension of β which is \bar{D}^{q-1} -exact for $t \neq 0$.

Conversely, if $\beta(t)$ is an extension of β such that

$$\beta(t) = \bar{D}^{q-1}\gamma(t)$$

for $t \neq 0$. Then $\gamma(t)$ can be chosen to be meromorphic in t with pole order $n \geq 1$ at $t = 0$. Let $\alpha_{n-1}(t) := t^n \gamma(t)$. Then $\alpha_{n-1}(t)$ is holomorphic in t and

$$O_n^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] = [j_0^n(\bar{D}^{q-1}(t^n \gamma(t))/t^n)] = [j_0^n(t^n \beta(t))/t^n] = [\beta].$$

This completes the proof. \square

Proposition 4.5. *Let $[j_0^{n-1}(\alpha_{n-1})(t)] \in H^{q-1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$ such that $O_n^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0$. Then there exist $n' \leq n$ and $[j_0^{n'-1}(\alpha_{n'-1})(t)] \in H^{q-1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^{n'})$ such that*

$$O_{n,n'-1}^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] = O_{n',n'-1}^{q-1}[j_0^{n'-1}(\alpha_{n'-1})(t)] \neq 0.$$

Proof. If $O_{n,n-1}^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0$, we can simply take $n' = n$ and $\alpha_{n'-1} = \alpha_{n-1}$. Otherwise, there exists α'_1 such that

$$\bar{D}_{n-1}^{q-1}\alpha'_1 = O_{n,n-1}^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)].$$

Then we have

$$O_{n-1,n-2}^{q-1}[\alpha'_1] = O_{n,n-2}^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)].$$

Since $O_n^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0$, we finally arrive at some n' such that

$$O_{n,n'-1}^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] = O_{n',n'-1}^{q-1}[j_0^{n'-1}(\alpha_{n'-1})(t)] \neq 0.$$

\square

These two propositions together prove the following

Corollary 4.6. *Every local extension of every non-exact element $\beta \in \ker(\bar{\partial}_E^q)$ is non-exact if and only if $O_{n,n-1}^{q-1} \equiv 0$ for all $n \geq 1$.*

Proof. For a fixed non-exact $\beta \in \ker(\bar{\partial}_E^q)$, if any extension of β is non-exact, then $[\beta] \notin \text{Im}(O_{n,n-1}^{q-1})$ for all $n \geq 1$. Hence $O_{n,n-1}^{q-1} \equiv 0$.

Conversely, if there is an extension of β such that it is exact for $t \neq 0$, then there exist $n \geq 1$ and $[j_0^{n-1}(\alpha_{n-1})(t)] \in H^{q-1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$ such that

$$O_n^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] = [\beta] \neq 0.$$

But we can also choose $n' \leq n$ and $[j_0^{n'-1}(\alpha_{n'-1})(t)] \in H^{q-1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^{n'})$ such that

$$O_{n,n'-1}^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] = O_{n',n'-1}^{q-1}[j_0^{n'-1}(\alpha_{n'-1})(t)] \neq 0.$$

This proves the corollary. \square

Lemma 4.7. *For each $q \geq 0$, $\pi_* \mathcal{D}^q$ is a flat \mathcal{O}_B -module.*

Proof. This follows from the fact that $(\pi_* \mathcal{D}^q)_t$ is torsion free and $\mathcal{O}_{B,t} \cong \mathbb{C}\{x-t\}$ is a PID for every $t \in B$. \square

We will need the following fact in homological algebra, whose proof can be found, e.g. in [3].

Proposition 4.8. *Let A be a Noetherian ring and C^\bullet be a finite cochain complex of flat A -modules whose cohomology $H^i(C^\bullet)$ is finitely generated for all i . Then there exists a cochain complex of finitely generated flat A -modules K^\bullet and a cochain map $C^\bullet \rightarrow K^\bullet$, which is a quasi-isomorphism. Moreover, for any A -module M , the natural map $C^\bullet \otimes M \rightarrow K^\bullet \otimes M$ is a quasi-isomorphism. Furthermore, if the dimension*

$$\dim_{k(\mathfrak{p})} H^q(K^\bullet \otimes k(\mathfrak{p}))$$

is locally constant in $\mathfrak{p} \in \text{Spec}(A)$, then for $i = q, q-1$, the δ -functors $T^i(M) := H^i(K^\bullet \otimes M)$ commute with base change.

We apply this proposition to the case $A = \mathcal{O}_{B,0}$, $C^\bullet = (\pi_* \mathcal{D}^\bullet)_0$ to prove the following

Proposition 4.9. *If $\dim_{k(t)} H^q((\pi_* \mathcal{D}^\bullet)_t \otimes k(t))$ is locally constant around $0 \in B$, then the canonical map*

$$H^q((\pi_* \mathcal{D}^\bullet)_0) \otimes k(0) \rightarrow H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0))$$

is an isomorphism.

Proof. Since $(\pi_* \mathcal{D}^\bullet)_0$ is a flat $\mathcal{O}_{B,0}$ -module, using Proposition 4.8, we obtain a complex of finitely generated flat $\mathcal{O}_{B,0}$ -modules K^\bullet such that

$$H^\bullet((\pi_* \mathcal{D}^\bullet)_0 \otimes M) \cong H^\bullet(K^\bullet \otimes M)$$

for any $\mathcal{O}_{B,0}$ -module M . We claim that the dimension

$$\dim_{k(\mathfrak{p})} H^q(K^\bullet \otimes k(\mathfrak{p}))$$

is locally constant in $\mathfrak{p} \in \text{Spec}(\mathcal{O}_{B,0})$.

First of all, since $\dim_{k(t)} H^q((\pi_* \mathcal{D}^\bullet)_t \otimes k(t))$ is locally constant, by Theorem 1.1 and Proposition 3.3, the sheaf $\mathcal{H}^q(\pi_* \mathcal{D}^\bullet) \cong R^q \pi_* \mathcal{E}$ is a locally free \mathcal{O}_B -module. Hence

$$H^q((\pi_* \mathcal{D}^\bullet)_0) \otimes k(0) \cong (R^q \pi_* \mathcal{E})_0 \otimes k(0) \cong H^q(X, E) \cong H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0)).$$

In particular

$$\begin{aligned} \dim_{k(0)} H^q(K^\bullet \otimes k(0)) &= \dim_{k(0)} H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0)) \\ &= \dim_{k(0)} H^q((\pi_* \mathcal{D}^\bullet)_0) \otimes k(0) \\ &= \dim_{k(0)} H^q(K^\bullet) \otimes k(0). \end{aligned}$$

Note that $\text{Spec}(\mathcal{O}_{B,0}) = \text{Spec}(\mathbb{C}\{x\}) = \{(0), (x)\}$. Let $Q := (\mathcal{O}_{B,0})_{(0)}$ be the localization of $\mathcal{O}_{B,0}$ at the ideal (0) , which is the field of quotients of $\mathcal{O}_{B,0}$. We obtain

$$H^q(K^\bullet \otimes k((0))) = H^q(K^\bullet \otimes Q) \cong H^q(K^\bullet) \otimes Q,$$

since localization is flat. On the other hand, as $(\mathcal{O}_{B,0})_{(x)} \cong \mathcal{O}_{B,0}$, we have

$$H^q(K^\bullet \otimes k((x))) \cong H^q(K^\bullet \otimes \mathcal{O}_{B,0}/m_0) = H^q(K^\bullet \otimes k(0)),$$

and so

$$\dim_{k((x))} H^q(K^\bullet \otimes k((x))) = \dim_{k(0)} H^q(K^\bullet \otimes k(0)) = \dim_{k(0)} H^q(K^\bullet) \otimes k(0).$$

As $H^q(K^\bullet) \cong H^q((\pi_* \mathcal{D}^\bullet)_0)$ is a free $\mathcal{O}_{B,0}$ -module and $\mathcal{O}_{B,0}$ is a local integral domain, we have

$$\dim_Q H^q(K^\bullet) \otimes Q = \dim_{k(0)} H^q(K^\bullet) \otimes k(0).$$

In summary, we conclude that

$$\dim_Q H^q(K^\bullet \otimes Q) = \dim_{k((x))} H^q(K^\bullet \otimes k((x))),$$

which means that $\dim_{k(\mathfrak{p})} H^q(K^\bullet \otimes k(\mathfrak{p}))$ is constant in $\mathfrak{p} \in \text{Spec}(\mathcal{O}_{B,0})$. Hence T^q commutes with base change. The required isomorphism now follows from taking $M = k(0)$ in Proposition 4.8. \square

Remark 4.10. By replacing $0 \in B$ by nearby $t \in B$, we note that the isomorphism holds in a neighborhood of 0.

We are now ready to prove our main result.

Theorem 4.11. $\dim_{k(t)} H^q(\mathcal{X}_t, \mathcal{E}_t)$ is locally constant if and only if $O_{m,m-1}^q \equiv 0$ and $O_{n,n-1}^{q-1} \equiv 0$ for all $m, n \geq 1$.

Proof. If $\dim_{k(t)} H^q(\mathcal{X}_t, \mathcal{E}_t) = \dim_{k(t)} H^q((\pi_* \mathcal{D}^\bullet)_t \otimes k(t))$ is locally constant, then Proposition 4.9 shows that the natural map

$$H^q((\pi_* \mathcal{D}^\bullet)_0) \otimes k(0) \rightarrow H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0))$$

is an isomorphism.

Now, let $c \in H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0))$. We can extend it to a nonzero local holomorphic section of $\mathcal{H}^q(\pi_* \mathcal{D}^\bullet)$ since $\mathcal{H}^q(\pi_* \mathcal{D}^\bullet)$ is locally free. Denote this extension by \tilde{c} . Consider the germ of this section $\tilde{c}_0 \in H^q((\pi_* \mathcal{D}^\bullet)_0)$. Choose a representative $\tilde{\alpha}_0 \in (\pi_* \mathcal{D}^\bullet)_0$ in this cohomology class. For each $m \geq 1$, maps $\tilde{\alpha}_0$ to $(\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^{m+1}$ via the quotient map p_m . Then the class $[p_m(\tilde{\alpha}_0)] \in H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^{m+1})$ is an m -th order extension of c . Hence $O_{m,m-1}^q \equiv 0$ by Proposition 4.2. Since m is arbitrary, $O_{m,m-1}^q \equiv 0$ for all $m \geq 1$.

For the obstruction map $O_{n,n-1}^{q-1}$, if $O_{n,n-1}^{q-1}[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0$ for some $n \geq 1$ and $[j_0^{n-1}(\alpha_{n-1})(t)] \in H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$, then we can find some nonzero $[\beta] \in H^q(\pi_* \mathcal{D}_0^\bullet \otimes k(0))$ and a local holomorphic extension $\tilde{\beta}$ of β such that it is exact only when $t \neq 0$. But $\mathcal{H}^q(\pi_* \mathcal{D}^\bullet)$ is locally free, any extension is locally nonzero by continuity. Therefore, $O_{n,n-1}^{q-1} \equiv 0$ for all $n \geq 1$ by Proposition 4.6.

Conversely, if both obstruction maps vanish, then for each $[\alpha] \in H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0))$, we obtain $\alpha(t) \in \Gamma(U, \pi_* \mathcal{D}^q)$ such that $\bar{D}_t \alpha(t) = 0$ in some neighborhood $U \subset B$ containing 0 and $[\alpha(0)] = [\alpha]$. Moreover, $\alpha(t)$ is non-exact since $O_{n,n-1}^{q-1} \equiv 0$ for all $n \geq 1$. Hence for fixed $t \in U$ we obtain an injective linear map

$$H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0)) \rightarrow H^q((\pi_* \mathcal{D}^\bullet)_t \otimes k(t)), \quad [\alpha] \mapsto [\alpha(t)].$$

Therefore,

$$\dim_{k(0)} H^q((\pi_* \mathcal{D}^\bullet)_0 \otimes k(0)) \leq \dim_{k(t)} H^q((\pi_* \mathcal{D}^\bullet)_t \otimes k(t)).$$

By upper semi-continuity, $\dim_{k(t)} H^q((\pi_* \mathcal{D}^\bullet)_t \otimes k(t)) = \dim_{k(t)} H^q(\mathcal{X}_t, \mathcal{E}_t)$ is locally constant. \square

Recall that by choosing a Hermitian metric on E and using the associated Chern connection, we can write

$$\bar{D}_t = \bar{\partial} + \varphi(t) \lrcorner \nabla + A(t),$$

where $\{(A(t), \varphi(t))\}_{t \in B}$ is the family of Maurer-Cartan elements which controls the deformations of (X, E) . Hence the n -th order obstruction maps $O_{n,n-1}^i : H^i((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n) \rightarrow H^{i+1}((\pi_* \mathcal{D}^\bullet)_0 \otimes \mathcal{O}_{B,0}/m_0^n)$, for $i = q, q-1$, defined in (1) can be rewritten as

$$(2) \quad O_{n,n-1}^i([\alpha_{n-1}]) = \left[t^{n-1} \sum_{j=0}^{n-1} (\varphi^{n-j} \lrcorner \nabla + A^{n-j}) \alpha_{n-1}^j \right].$$

as claimed in Theorem 1.2.

Example 4.12. We first consider the case when $E = T_X$, the holomorphic tangent bundle of X . We deform the pair (X, T_X) to (X_t, T_{X_t}) , where T_{X_t} is the holomorphic tangent bundle to X_t (note that T_X may have other deformations which are not isomorphic to the holomorphic tangent bundle on X_t). In this case, the $\text{End}(T_X)$ -part of the Maurer-Cartan element $(A(t), \varphi(t))$ is given by

$$A(t) = -T(\varphi(t), \bullet) - \nabla_\bullet \varphi(t),$$

where $T : \Omega^{0,\bullet}(T_X) \times \Omega^{0,\bullet}(T_X) \rightarrow \Omega^{0,\bullet}(T_X)$ is the graded torsion on T_X defined by

$$T(\varphi, \psi) := \varphi \lrcorner \nabla \psi - (-1)^{|\varphi||\psi|} \psi \lrcorner \nabla \varphi - [\varphi, \psi].$$

So we have

$$\bar{D}_t^\bullet = \bar{\partial}_{T_X}^\bullet + [\varphi_t, -].$$

For $\alpha_{n-1} \in \Omega^{0,q}(T_X) \otimes \mathcal{O}_{B,0}$ such that $\bar{D}_t^q \alpha_{n-1} = 0 \bmod t^{n-1}$, we have

$$t^{n-1} (j_0^n (\bar{D}_t^q \alpha_{n-1}) / t^n) = t^{n-1} \left(\bar{\partial}_{T_X} \alpha_{n-1}^n + \sum_{j=0}^{n-1} [\varphi^{n-j}, \alpha_{n-1}^j] \right).$$

As a class in $H^{q+1}(\Omega^{0,\bullet}(T_X) \otimes \mathcal{O}_{B,0}/m_0^n, \bar{D}_{n-1}^\bullet)$, it is equal to

$$\left[t^{n-1} \sum_{j=0}^{n-1} [\varphi^{n-j}, \alpha_{n-1}^j] \right].$$

Hence the obstruction is given by

$$O_{n,n-1}^q[j_0^{n-1}(\alpha_{n-1})(t)] = \left[t^{n-1} \sum_{j=0}^{n-1} [\varphi^{n-j}, \alpha_{n-1}^j] \right].$$

Example 4.13. For the case $E = T_X^*$, we have

$$\bar{D}_t^\bullet = \bar{\partial}_{T_X^*}^\bullet + [\varphi_t, -]^*,$$

where $[\varphi_t, -]^* : \Omega^{0,q}(T_X^*) \rightarrow \Omega^{0,q+1}(T_X^*)$ is given by

$$[\varphi_t, \eta]^*(v) := [\varphi_t, \eta(v)] - (-1)^q \eta([\varphi_t, v]) = \varphi_t \lrcorner \partial(\eta(v)) - (-1)^q \eta([\varphi_t, v])$$

for $v \in \Omega^0(T_X)$. Since

$$\varphi_t \lrcorner \partial(\eta(v)) - (-1)^q \eta([\varphi_t, v]) = (\varphi_t \lrcorner \partial \eta)(v) + v \lrcorner \partial(\varphi_t \lrcorner \eta),$$

the obstruction is given by

$$O_{n,n-1}^q[j_0^{n-1}(\alpha_{n-1})(t)] = \left[t^{n-1} \sum_{j=0}^{n-1} (\varphi^{n-j} \lrcorner \partial \alpha_{n-1}^j + \partial(\varphi^{n-j} \lrcorner \alpha_{n-1}^j)) \right].$$

For $E = \wedge^q T_X^*$, we have

$$\begin{aligned} \bar{D}_t(\alpha_1 \wedge \cdots \wedge \alpha_p) &= \sum_{j=1}^{p-1} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge \bar{D}_t \alpha_j \wedge \cdots \wedge \alpha_p \\ &= \bar{\partial}(\alpha_1 \wedge \cdots \wedge \alpha_p) + \sum_{j=1}^p (-1)^{j-1} \alpha_1 \wedge \cdots \wedge [\varphi_t, \alpha_j]^* \wedge \cdots \wedge \alpha_p, \end{aligned}$$

where $\alpha_j \in \Omega^0(T_X^*)$. Then

$$\begin{aligned} &\sum_{j=1}^{p-1} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge (\varphi_t \lrcorner \partial \alpha_j) \wedge \cdots \wedge \alpha_p + \sum_{j=1}^{p-1} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge \partial(\varphi_t \lrcorner \alpha_j) \wedge \cdots \wedge \alpha_p \\ &= \varphi_t \lrcorner (\partial(\alpha_1 \wedge \cdots \wedge \alpha_p)) + \partial(\varphi_t \lrcorner (\alpha_1 \wedge \cdots \wedge \alpha_p)). \end{aligned}$$

Hence the obstruction map is given by

$$O_{n,n-1}^q[j_0^{n-1}(\alpha_{n-1})(t)] = \left[t^{n-1} \sum_{j=0}^{n-1} (\varphi^{n-j} \lrcorner \partial \alpha_{n-1}^j + \partial(\varphi^{n-j} \lrcorner \alpha_{n-1}^j)) \right].$$

These two examples recover the obstruction formulae in [7, 8].

APPENDIX A. CONVERGENCE

Consider an element $\alpha \in \ker(\bar{\partial}^q)$. Suppose that the obstruction maps $O_{n,n-1}^q$ vanish for all $n \geq 1$. Then we obtain a formal extension $\alpha(t)$ of α , that is, as a formal power series in $\Omega^{0,q}(E)$,

$$\bar{D}_t^q \alpha(t) = 0.$$

In this appendix, we show that one can always choose an extension $\alpha(t)$ with nonzero radius of convergent. To achieve this, we shall work on the Kuranishi family of (X, E) [6], following the approach of the book [5].

We choose a hermitian metric for E and consider the equation

$$\alpha(t) + \bar{\partial}_E^* G_E(\varphi(t) \lrcorner \nabla + A(t)) \alpha(t) = 0, \quad \alpha(0) = \alpha \in \ker(\bar{\partial}_E^q),$$

with $\alpha(t)$ holomorphic in the variable t . Then $\alpha(t)$ can be solved by the recursive relations:

$$\alpha^n + \sum_{i=0}^{n-1} \bar{\partial}_E^* G_E(\varphi_{n-i} \lrcorner \nabla + A_{n-i}) \alpha^i = 0, \quad n \geq 1.$$

We shall prove that $\alpha(t) := \sum_{n=0}^{\infty} \alpha^n t^n$ converges uniformly in the Hölder norm $\|\cdot\|_{k+\alpha}$. First of all, let us recall the obvious estimates

$$\begin{aligned} \|[(A, \varphi), (B, \psi)]\|_{k+\alpha} &\leq C_{k,\alpha} \|(A, \varphi)\|_{k+\alpha+1} \|(B, \psi)\|_{k+\alpha+1}, \\ \|(\varphi \lrcorner \nabla + A)\delta\|_{k+\alpha} &\leq C'_{k,\alpha} \|(A, \varphi)\|_{k+\alpha+1} \|\delta\|_{k+\alpha+1} \end{aligned}$$

for any $(A, \varphi), (B, \psi) \in \Omega^\bullet(\mathcal{E})$ and $\delta \in \Omega^{0,\bullet}(E)$, where $C_{k,\alpha}, C'_{k,\alpha}$ are positive constants which depend only on k, α . We may assume that $C'_{k,\alpha}$ is larger so that

$$\begin{aligned} \|[(A, \varphi), (B, \psi)]\|_{k+\alpha} &\leq C_{k,\alpha} \|(A, \varphi)\|_{k+\alpha} \|(B, \psi)\|_{k+\alpha}, \\ \|(\varphi \lrcorner \nabla + A)\delta\|_{k+\alpha} &\leq C_{k,\alpha} \|(A, \varphi)\|_{k+\alpha} \|\delta\|_{k+\alpha} \end{aligned}$$

for any $(A, \varphi), (B, \psi) \in \Omega^\bullet(A(E))$ and $\delta \in \Omega^{0,\bullet}(E)$. Next, we have the estimates

$$\begin{aligned} \|\bar{\partial}_E^* G_E \delta\|_{k+\alpha} &\leq \tilde{C}_{k,\alpha} \|\delta\|_{k-1+\alpha}, \\ \|\bar{\partial}_{A(E)}^* G_{A(E)}(A, \varphi)\|_{k+\alpha} &\leq \tilde{C}'_{k,\alpha} \|(A, \varphi)\|_{k-1+\alpha} \end{aligned}$$

for all $(A, \varphi) \in \Omega^{0,\bullet}(A(E))$ and $\delta \in \Omega^{0,\bullet}(E)$, where $G_{A(E)}, G_E$ are Green's operators correspond to $A(E), E$, respectively, and $\tilde{C}_{k,\alpha}, \tilde{C}'_{k,\alpha}$ are positive constants depending only on k, α . Again we assume that $\tilde{C}_{k,\alpha}$ is larger.

Proposition A.1. *For $|t|$ small, $\alpha(t) = \sum_{n=0}^{\infty} \alpha^n t^n$ converges in the norm $\|\cdot\|_{k+\alpha}$ and $\alpha(t)$ is a smooth solution.*

Proof. The proof is rather standard, and we follow the book [5] very closely.

First we observe that $\delta(t) := t \cdot \alpha(t)$ also satisfies the equation

$$\delta(t) + \bar{\partial}_E^* G_E((\varphi(t) \lrcorner \nabla + A(t))\delta(t)) = 0.$$

Denote $\delta_n(t) = \delta(t) \bmod t^{n+1}$ (similar meaning for $A^n(t)$ and $\varphi^n(t)$). Let

$$B(t) := \frac{\beta}{16\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n^2} t^n := \sum_{n=1}^{\infty} B^n t^n,$$

where β, γ are positive constants which are to be chosen. We want to choose β, γ such that $\|\delta^n\|_{k+\alpha} \leq B^n$ for all $n \geq 1$ (this condition will be denoted by $\|\delta_n(t)\|_{k+\alpha} \ll B(t)$). This is of course possible for $n = 1$. Hence we assume that this is possible up to order $n - 1$, for some $n > 1$.

For any (A, φ) and δ , we have

$$\|\bar{\partial}_E^* G_E((\varphi \lrcorner \nabla + A)\delta)\|_{k+\alpha} \leq \tilde{C}_{k,\alpha} \|(\varphi \lrcorner \nabla + A)\delta\|_{k-1+\alpha} \leq \tilde{C}_{k,\alpha} C_{k,\alpha} \|(A, \varphi)\|_{k+\alpha} \|\delta\|_{k+\alpha},$$

so the induction hypothesis gives

$$\|\delta_n(t)\|_{k+\alpha} \leq \tilde{C}_{k,\alpha} C_{k,\alpha} \|(A^n(t), \varphi^n(t))\|_{k+\alpha} \|\delta_{n-1}(t)\|_{k+\alpha} \ll \tilde{C}_{k,\alpha} C_{k,\alpha} \|(A^n(t), \varphi^n(t))\|_{k+\alpha} B(t).$$

It follows from Proposition 2.4, p.162 in [5] that, when β, γ are chosen such that

$$\tilde{C}_{k,\alpha} C_{k,\alpha} \frac{\beta}{\gamma} < 1 \quad \text{and} \quad \|(A^1(t), \varphi^1(t))\|_{k+\alpha} \ll B(t),$$

we have $\|(A^n(t), \varphi^n(t))\|_{k+\alpha} \ll B(t)$ for any $n \geq 1$. Hence

$$\|\delta_n(t)\|_{k+\alpha} \ll \tilde{C}_{k,\alpha} C_{k,\alpha} (B(t))^2.$$

It can also be proved (see Lemma 3.6, p. 50 in [5]) that

$$(B(t))^2 \ll \frac{\beta}{\gamma} B(t).$$

Therefore, for the above choices of β, γ , we have

$$\|\delta_n(t)\|_{k+\alpha} \ll B(t).$$

Since $B(t)$ converges on $|t| < \gamma^{-1}$, we see that $\delta(t)$, and hence $\alpha(t)$, also converges there.

Finally, $\alpha(t)$ satisfies

$$\left(\frac{\partial^2}{\partial t \partial \bar{t}} + \Delta_E + \bar{\partial}_E^*(\varphi(t) \lrcorner \nabla + A(t)) \right) \alpha(t) = 0.$$

Since the operator

$$\frac{\partial^2}{\partial t \partial \bar{t}} + \Delta_E + \bar{\partial}_E^*(\varphi(t) \lrcorner \nabla + A(t))$$

is elliptic for $|t|$ small, regularity guarantees smoothness of $\alpha(t)$. \square

Next we have the following

Proposition A.2. *The $\alpha(t)$ defined above satisfies*

$$\bar{D}_t \alpha(t) = (\bar{\partial}_E + \varphi(t) \lrcorner \nabla + A(t)) \alpha(t) = 0$$

if and only if $\mathbb{H}((\varphi(t) \lrcorner \nabla + A(t)) \alpha(t)) = 0$.

Proof. If $\bar{D}_t \alpha(t) = 0$, then it is clear that $\mathbb{H}((\varphi(t) \lrcorner \nabla + A(t)) \alpha(t)) = 0$ since $\mathbb{H} \bar{\partial}_E = 0$.

Conversely, suppose that $\mathbb{H}((\varphi(t) \lrcorner \nabla + A(t)) \alpha(t)) = 0$. Let $\psi(t) := \bar{D}_t \alpha(t)$. Since $\alpha(t)$ satisfies

$$\alpha(t) + \bar{\partial}_E^* G_E(\varphi(t) \lrcorner \nabla + A(t)) \alpha(t) = 0,$$

applying $\bar{\partial}_E$ gives

$$\bar{\partial}_E \alpha(t) = -\bar{\partial}_E \bar{\partial}_E^* G_E(\varphi(t) \lrcorner \nabla + A(t)) \alpha(t).$$

Then

$$\psi(t) = -\bar{\partial}_E \bar{\partial}_E^* G_E(\varphi(t) \lrcorner \nabla + A(t)) \alpha(t) + (\varphi(t) \lrcorner \nabla + A(t)) \alpha(t).$$

Since $(\varphi(t) \lrcorner \nabla + A(t)) \alpha(t)$ has no harmonic part, we have

$$\begin{aligned} \psi(t) &= \bar{\partial}_E^* \bar{\partial}_E G_E(\varphi(t) \lrcorner \nabla + A(t)) \alpha(t) \\ &= \bar{\partial}_E^* G_E[(\bar{\partial}_{T_X} \varphi(t) \lrcorner \nabla + \varphi(t) \lrcorner F \nabla + \bar{\partial}_{\text{End}(E)} A(t)) \alpha(t) - (\varphi(t) \lrcorner \nabla + A(t)) \bar{\partial}_E \alpha(t)] \\ &= \bar{\partial}_E^* G_E \left[-\frac{1}{2} [(A(t), \varphi(t)), (A(t), \varphi(t))] \cdot \alpha(t) - (\varphi(t) \lrcorner \nabla + A(t)) (\psi(t) - (\varphi(t) \lrcorner \nabla + A(t)) \alpha(t)) \right] \\ &= -\bar{\partial}_E^* G_E[(\varphi(t) \lrcorner \nabla + A(t)) \psi(t)], \end{aligned}$$

where the Lie bracket acts by

$$[(A(t), \varphi(t)), (A(t), \varphi(t))] \cdot \alpha(t) := (2\varphi(t) \lrcorner \nabla + [A(t), A(t)] + [\varphi(t), \varphi(t)] \lrcorner \nabla) \alpha(t).$$

and we have used

$$((\varphi(t) \lrcorner \nabla + A(t))^2, \frac{1}{2} [\varphi, \varphi]) = \frac{1}{2} [(A(t), \varphi(t)), (A(t), \varphi(t))]$$

in the last equality. By the previous estimates, we have

$$\|\psi(t)\|_{k+\alpha} \leq \tilde{C}_{k,\alpha} C_{k,\alpha} \|\psi(t)\|_{k+\alpha} \|(A(t), \varphi(t))\|_{k+\alpha},$$

For $|t|$ small enough, we always have $\tilde{C}_{k,\alpha} C_{k,\alpha} \|(A(t), \varphi(t))\|_{k+\alpha} < 1$. Therefore, $\psi(t) = 0$. This completes the proof. \square

Finally, we claim that the harmonic part of $(\varphi(t) \lrcorner \nabla + A(t))\alpha(t)$ vanishes under the assumption that $O_{n,n-1}^q \equiv 0$ for all $n \geq 1$.

Proposition A.3. *The obstructions $O_{n,n-1}^q \equiv 0$ for all $n \geq 1$ if and only if for any $\alpha(t)$ satisfying*

$$\alpha(t) + \bar{\partial}_E^* G_E(\varphi(t) \lrcorner \nabla + A(t))\alpha(t) = 0$$

and $\bar{\partial}_E \alpha(0) = 0$, we have $\bar{D}_t \alpha(t) = 0$.

Proof. If $\mathbb{H}((\varphi(t) \lrcorner \nabla + A(t))\alpha(t)) = 0$ for any $\alpha = \alpha(0) \in \ker(\bar{\partial}_E^q)$, then $\alpha(t)$ is an extension of α . Hence $O_{n,n-1}^q \equiv 0$ for all $n \geq 1$.

For the converse direction, we proceed by induction on n . For $n = 1$, we have

$$j_0^1(\bar{D}_t \alpha_0)(t) = \bar{D}_0(j_0^0(\beta)(t)) = j_0^0(\bar{D}_t \beta)(t)$$

for some local section $\beta = \sum_{n=0}^{\infty} \beta_n t^n$, i.e.

$$(\varphi_1 \lrcorner \nabla + A_1)\alpha_0 = \bar{\partial}_E \beta_0.$$

Hence $\mathbb{H}((\varphi_1 \lrcorner \nabla + A_1)\alpha_0) = 0$. Assume that $\mathbb{H}((\varphi(t) \lrcorner \nabla + A(t))\alpha(t))$ vanishes up to order $n-1$. Then $\alpha(t)$ defines an $(n-1)$ -st order extension of α and by assumption $O_{n,n-1}[j_0^{n-1}(\alpha(t))] = 0$. Therefore,

$$t^{n-1} \sum_{j=0}^{n-1} (\varphi_{n-j} \lrcorner \nabla + A_{n-j})\alpha^j = \bar{D}_{n-1}(j_0^{n-1}(\beta)(t)) = j_0^{n-1}(\bar{D}_t \beta)(t).$$

Hence

$$\sum_{j=0}^{n-1} (\varphi_{n-j} \lrcorner \nabla + A_{n-j})\alpha^j = \bar{\partial}_E \beta^{n-1} + \sum_{j=0}^{n-2} (\varphi_{n-1-j} \lrcorner \nabla + A_{n-1-j})\beta^j$$

and

$$\bar{\partial}_E \beta^k + \sum_{j=0}^{k-1} (\varphi_{k-j} \lrcorner \nabla + A_{k-j})\beta^j = 0$$

for $k \leq n-2$.

The last $(n-1)$ -equations simply means that β defines an extension of β_0 of order $n-2$. By assumption, we have $O_{n-1,n-2}^q[j_0^{n-2}(\beta)(t)] = 0$, and so

$$\sum_{j=0}^{n-2} (\varphi_{n-1-j} \lrcorner \nabla + A_{n-1-j})\beta^j = \bar{\partial}_E \gamma^{n-2} + \sum_{j=0}^{n-3} (\varphi_{n-2-j} \lrcorner \nabla + A_{n-2-j})\gamma^j$$

for some $\gamma = \sum_{n=0}^{\infty} \gamma_n t^n$. Keep repeating the previous argument, this reduces to the $n = 1$ case and so

$$\sum_{j=0}^{n-1} (\varphi_{n-j} \lrcorner \nabla + A_{n-j})\alpha^j$$

is $\bar{\partial}_E$ -exact and therefore, has no harmonic part. This completes the induction argument \square

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